B.Sc. part - III paper V

(1) Continuity ~~and differentiability~~ of function

Def:- Let $f(x,y)$ be real valued function of two variables x and y defined on set $S \subset R^2$ and $(a,b) \in S$. The function f is said to be continuous at (a,b) if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x-a| < \delta, \; |y-b| < \delta \Rightarrow |f(x,y) - f(a,b)| < \epsilon$$

for all $(x,y) \in S$

in other words $f(x,y)$ is said to be continuous at (a,b) if simultaneous limit $\lim_{\substack{x \to a \\ y \to b}} f(x,y)$ exists and is equal to its functional value $f(a,b)$ at (a,b)

problem (1) Investigate the

(2) Continuity at $(0,0)$ of

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Solution:- Here

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}, (x,y) \neq (0,0)$$

and $f(0,0)=0$

To test continuity at the origin, we find that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \quad -(1)$$

If we put $x = my$ in (1) we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{m^2y^2-y^2}{m^2y^2+y^2}$$

$$= \lim_{y\to 0} \frac{m^2-1}{m+1}$$

which depend upon m.

Hence the function is not continuous

problem ②

(3) Examine whether the function

$$f(x,y) = \begin{cases} x^2+4y & ,(x,y) \neq (1,2) \\ 0 & ,(x,y) = (1,2) \end{cases}$$

is continuous at $(1,2)$

Soln:- Here $f(x,y) = \begin{cases} x^2+4y \\ 0 \end{cases}$

$$\lim_{\substack{x\to 1 \\ y\to 2}} (x^2+4y) = 1^2+4\times 2$$
$$= 1+8 = 9 \text{, so the}$$

limit exists and equal to 9

Since $f(1,2)=0$ and $\lim_{\substack{x\to 1 \\ y\to 2}} f(x,y) = 9$

we have $\lim_{\substack{x\to 1 \\ y\to 2}} f(x,y) \neq f(1,2)$

Hence the function is not continuous at $(1,2)$

problem (5) investigate the continuity at $(0,0)$ of

(4) $$f(x,y) = \begin{cases} 2xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Soln:- Here $f(x,y) = 2xy\dfrac{x^2-y^2}{x^2+y^2}$ where $(x,y) \neq (0,0)$

and $f(x,y) = 0$ when $(x,y) = (0,0)$

putting $x = r\cos\theta$, $y = r\sin\theta$ we have

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) \Rightarrow r^2 = x^2+y^2$

Now given $\epsilon > 0$ we have

$$\left|2xy\frac{x^2-y^2}{x^2+y^2}\right| = \left|2r^2\cos\theta\sin\theta \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2(\sin^2\theta + \cos^2\theta)}\right|$$

$$= \left|2r^2\cos\theta\sin\theta \frac{r^2\cos 2\theta}{r^2}\right|$$

$$= \left|r^2\sin 2\theta\cos 2\theta\right| = \left|\frac{r^2}{2}\sin 4\theta\right|$$

$$\leq \frac{r^2}{2} = \frac{x^2+y^2}{2} < \epsilon$$

Thus for given $\epsilon > 0$ there exists $\delta > 0$ s.t. $\left|2xy\dfrac{x^2-y^2}{x^2+y^2} - 0\right| < \epsilon$ for $|x-0| < \delta$, $|y-0| < \delta$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

He...

problem (4) construct an example of a function which is separetly
(5) continuous but not continuous
soln:- Let a function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & \text{at } (0,0) \end{cases}$$

This function is not continuous at (0,0) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists

we see by taking $(x,y) \to (0,0)$ through the curve $y = mx, m \neq 0$

we $f(x,y) = \frac{x^2 \cdot m^2x^2}{x^4+m^4x^4} = \frac{m^2x^4}{x^4(1+m^4)}$

$= \frac{m^2}{1+m^4}$

which is depend on m

But $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2 \cdot 0}{x^4 + 0} = 0$

$= f(0,0)$

and $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0 \cdot y^2}{0+y^4} = 0$

$= f(0,0)$

Thus f is separately continuous but it is not continuous at (0,0)